

THE CONTINUITY OF DELIGNE'S PAIRING

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INTRODUCTION

In the paper [11, §1.2], S. Zhang expected the metric of Deligne's pairing to be continuous. In this note, we will give an affirmative answer for his question. Namely, we will show the following theorem.

Theorem A. *Let $f : X \rightarrow S$ be a flat and projective morphism of algebraic varieties over \mathbb{C} with $n = \dim f$. Let $\bar{L}_0, \dots, \bar{L}_n$ be C^∞ -hermitian line bundles on X . Then, the metric of Deligne's pairing $\langle \bar{L}_0, \dots, \bar{L}_n \rangle(X/S)$ is continuous.*

As an application of the above theorem, we have the following result, which is, in some sense, a generalization of Kawaguchi's Hodge index theorem [6].

Theorem B. *Let K be a finitely generated field over \mathbb{Q} with $d = \text{tr. deg}_{\mathbb{Q}}(K)$, and let $\bar{B} = (B, \bar{H})$ be a polarization of K (for details, see [8]). Let X be a smooth projective curve of genus g over K , J_X the Jacobian of X , and let Θ_X be a symmetric theta divisor on J_X . For a line bundle L on X with $\deg(L) = 0$, there is a sequence of C^∞ -models $(\mathcal{X}_m, \bar{\mathcal{L}}_m)$ of (X, L) over B with*

$$\lim_{n \rightarrow \infty} \widehat{\deg}(\hat{c}_1(\bar{\mathcal{L}}_m) \cdot \hat{c}_1(\bar{\mathcal{L}}_m) \cdot \hat{c}_1(f_m^*(\bar{H}))^d) = -2\hat{h}_{\Theta_X}^{\bar{B}}([L]),$$

where f_m is the canonical morphism $\mathcal{X}_m \rightarrow B$.

Finally, we would like to express hearty thanks to Prof. Kawaguchi for telling us the exact formula (3.5). The author also thanks Prof. Zhang for his nice comments.

1. DELIGNE'S PAIRING WITH METRIC

Here we give a quick review of Deligne's pairing with metric. For details, see [1, §8.1], [2], [3], [4], [5], [11, §1.1-§1.2] and etc.

Let $f : X \rightarrow S$ be a flat and projective morphism of integral schemes of relative dimension n . Then, we have Deligne's pairing

$$\langle \dots, \rangle(X/S) : \overbrace{\text{Pic}(X) \times \dots \times \text{Pic}(X)}^{(n+1)\text{-times}} \rightarrow \text{Pic}(S).$$

Roughly speaking, Deligne's pairing is given by

$$c_1(\langle L_0, \dots, L_n \rangle(X/S)) = f_*(c_1(L_0) \cdots c_1(L_n))$$

for $L_0, \dots, L_n \in \text{Pic}(X)$. Note that if L_0, \dots, L_n have local (with respect to S) sections l_0, \dots, l_n without intersections, then the symbol $\langle l_0, \dots, l_n \rangle$ gives rise to a local base of $\langle L_0, \dots, L_n \rangle(X/S)$.

Next we assume that X and S are defined over \mathbb{C} . Let $\overline{L}_0, \dots, \overline{L}_n$ be C^∞ -hermitian line bundles on X . The metric of $\langle L_0, \dots, L_n \rangle(X/S)$ induced by metrics of $\overline{L}_0, \dots, \overline{L}_n$ is defined inductively in the following way. Let l_0, \dots, l_n be local sections of L_0, \dots, L_n such that l_0, \dots, l_n have no intersections and $Y = \text{div}(l_n)$ is integral and flat over S . Then, the length of $\langle l_0, \dots, l_n \rangle$ is given by

$$\|\langle l_0|_Y, \dots, l_{n-1}|_Y \rangle\| \exp \left(\int_{X/S} \log \|l_n\| \bigwedge_{i=0}^{n-1} c_1(\overline{L}_i) \right),$$

where

$$\left(\int_{X/S} \log \|l_n\| \bigwedge_{i=0}^{n-1} c_1(\overline{L}_i) \right) (s) = \int_{X_s} \log \|l_n\| \bigwedge_{i=0}^{n-1} c_1(\overline{L}_i)$$

for each $s \in S(\mathbb{C})$. We denote the line bundle $\langle L_0, \dots, L_n \rangle(X/S)$ with the above metric by $\langle \overline{L}_0, \dots, \overline{L}_n \rangle(X/S)$. For simplicity,

$$\overbrace{\langle \overline{L}_0, \dots, \overline{L}_0 \rangle}^{a_0\text{-times}} \overbrace{\langle \overline{L}_1, \dots, \overline{L}_1 \rangle}^{a_1\text{-times}} \dots \overbrace{\langle \overline{L}_t, \dots, \overline{L}_t \rangle}^{a_t\text{-times}}(X/S)$$

is denoted by $\langle \overline{L}_0^{a_0}, \overline{L}_1^{a_1}, \dots, \overline{L}_t^{a_t} \rangle(X/S)$.

2. THE CONTINUITY OF THE FIBER INTEGRAL

In this section, we will consider the continuity of the fiber integral of C^∞ -forms. First of all, let us fix the C^∞ of hermitian line bundles in this note, which coincides with the sense of [11] and [8].

Let X be an algebraic variety over \mathbb{C} , and \overline{L} a continuous hermitian line bundle on X . We say \overline{L} is a C^∞ -hermitian line bundle if, for any complex manifolds M and any analytic maps $f : M \rightarrow X$, $f^*(\overline{L})$ is a C^∞ -hermitian line bundle on M . In the same way, a continuous function ϕ on X is said to be C^∞ if, for any complex manifolds M and any analytic maps $f : M \rightarrow X$, $f^*(\phi)$ is a C^∞ -function on M .

The proof of the following theorem is the main purpose of this section.

Theorem 2.1. *Let $f : X \rightarrow S$ be a flat and projective morphism of algebraic varieties over \mathbb{C} with $n = \dim f$. Let $\overline{L}_1, \dots, \overline{L}_n$ be C^∞ -hermitian line bundles on X , and let ϕ be a C^∞ -function on X . Then, the fiber integral $\int_{X/S} \phi \bigwedge_{i=1}^n c_1(\overline{L}_i)$ is continuous, namely, the function given by*

$$S(\mathbb{C}) \ni s \mapsto \int_{X_s} \phi \bigwedge_{i=1}^n c_1(\overline{L}_i)$$

is continuous.

Proof. Before starting the proof of our theorem, we would like to introduce the C^∞ of hermitian line bundles in strong sense. A continuous function ϕ on an algebraic variety X over \mathbb{C} is said to be *strongly C^∞* at $x \in X$ if there are an open neighborhood U of x , a complex manifold V , and a C^∞ -function Φ on V such that U is a closed analytic subset of V and $\phi = \Phi|_U$. If ϕ is strongly C^∞ at any points of X , then ϕ is said to be *strongly C^∞* on X . We say a continuous hermitian line

bundle $\overline{L} = (L, \|\cdot\|)$ is *strongly* C^∞ if, for any local basis l of L on any open set, $\|l\|$ is strongly C^∞ around there.

First, let us consider the following claim.

Claim 2.1.1. *There is a commutative diagram:*

$$\begin{array}{ccc} X & \xleftarrow{v} & X' \\ f \downarrow & & \downarrow f' \\ S & \xleftarrow{u} & S' \end{array}$$

with the following properties.

- (1) $f' : X' \rightarrow S'$ is a flat and projective morphism of algebraic varieties over \mathbb{C} , and S' is non-singular.
- (2) u and v are projective birational morphisms.
- (3) $v^*(\overline{L}_1), \dots, v^*(\overline{L}_n)$ and $v^*(\phi)$ are strongly C^∞ .

Let $\mu : X_1 \rightarrow X$ be a resolution of singularities of X . Note that we may take μ as a projective morphism. Let $f_1 : X_1 \rightarrow Y$ be the composition of morphisms $X_1 \xrightarrow{\mu} X \xrightarrow{f} S$. Then, by applying Raynaud's flattening theorem [9] to $f_1 : X_1 \rightarrow S$, we have the following commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\mu} & X_1 & \xleftarrow{v_1} & X' \\ f \downarrow & & f_1 \downarrow & & \downarrow f' \\ S & \xlongequal{\quad} & S & \xleftarrow{u} & S' \end{array}$$

where u is a projective birational morphism of algebraic varieties over \mathbb{C} , $f' : X' \rightarrow S'$ is a flat and projective morphism of algebraic schemes over \mathbb{C} , and X' is the main part of $X_1 \times_S S'$. If S' is not non-singular, taking a desingularization $S'' \rightarrow S'$ of S' and replacing X' by $X' \times_{S'} S''$, we may assume that S' is non-singular. Since X is integral, so is the generic fiber of $f : X \rightarrow S$. Thus, by virtue of Lemma 4.2, we can see that X' is an algebraic variety over \mathbb{C} . In particular, v_1 is birational. Let v be the composition of morphisms $X' \xrightarrow{v_1} X_1 \xrightarrow{\mu} X$. Then, we have our desired commutative diagram satisfying (1) and (2). Moreover, since X_1 is non-singular, $\mu^*(\overline{L}_1), \dots, \mu^*(\overline{L}_n)$ and $\mu^*(\phi)$ are strongly C^∞ . Thus, so are $v^*(\overline{L}_1), \dots, v^*(\overline{L}_n)$ and $v^*(\phi)$.

Let us go back to the proof of our theorem. The commutative diagram in the above claim gives rise to the following commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \times_S S' & \xleftarrow{\pi} & X' \\ f \downarrow & & \downarrow p_2 & & \downarrow f' \\ S & \xleftarrow{u} & S' & \xlongequal{\quad} & S' \end{array}$$

where p_i ($i = 1, 2$) is the projection to the i -th factor, and $v = p_1 \cdot \pi$. Note that by Lemma 4.2, $X \times_S S'$ is integral. Since the fiber $(X \times_S S')_{s'}$ over s' is canonically isomorphic to the fiber $X_{u(s')}$

over $u(s')$ for each $s' \in S'(\mathbb{C})$, we can see

$$\int_{(X \times_S S')_{s'}} p_1^*(\phi) \bigwedge_{i=1}^n c_1(p_1^*(\bar{L}_i)) = \int_{X_{u(s')}} \phi \bigwedge_{i=1}^n c_1(\bar{L}_i),$$

which shows us

$$(2.1.2) \quad \int_{X \times_S S'/S'} p_1^*(\phi) \bigwedge_{i=1}^n c_1(p_1^*(\bar{L}_i)) = u^* \left(\int_{X/S} \phi \bigwedge_{i=1}^n c_1(\bar{L}_i) \right).$$

On the other hand, we would like to see

$$(2.1.3) \quad \int_{X'_{s'}} v^*(\phi) \bigwedge_{i=1}^n c_1(v^*(\bar{L}_i)) = \int_{(X \times_S S')_{s'}} p_1^*(\phi) \bigwedge_{i=1}^n c_1(p_1^*(\bar{L}_i))$$

for all $s' \in S'(\mathbb{C})$. For this purpose, considering a general curve passing through s' , we may assume that $\dim S = 1$. Then, $X'_{s'}$ and $(X \times_S S')_{s'}$ are Cartier divisors and $\pi^*((X \times_S S')_{s'}) = X'_{s'}$. Thus, $\pi_*(X'_{s'}) = (X \times_S S')_{s'}$ as cycles by virtue of the projection formula and the fact that π is birational. Here we recall the projection formula of integral version:

Formula 2.1.4. *Let $g : V \rightarrow T$ be a surjective morphism of complete algebraic varieties over \mathbb{C} , $\bar{L}_1, \dots, \bar{L}_{\dim V}$ C^∞ -hermitian line bundles on T , and let ϕ be a C^∞ -function on T . Then, we have*

$$\int_V g^*(\phi) \bigwedge_{i=1}^{\dim V} c_1(g^*(\bar{L}_i)) = \begin{cases} [\mathbb{C}(V) : \mathbb{C}(T)] \int_T \phi \bigwedge_{i=1}^{\dim V} c_1(\bar{L}_i) & \text{if } \dim V = \dim T, \\ 0 & \text{if } \dim V > \dim T. \end{cases}$$

Hence, by using the above projection formula,

$$\int_{X'_{s'}} v^*(\phi) \bigwedge_{i=1}^n c_1(v^*(\bar{L}_i)) = \int_{\pi_*(X'_{s'})} p_1^*(\phi) \bigwedge_{i=1}^n c_1(p_1^*(\bar{L}_i)) = \int_{(X \times_S S')_{s'}} p_1^*(\phi) \bigwedge_{i=1}^n c_1(p_1^*(\bar{L}_i)).$$

Therefore, gathering (2.1.2) and (2.1.3), we get

$$\int_{X'/S'} v^*(\phi) \bigwedge_{i=1}^n c_1(v^*(\bar{L}_i)) = u^* \left(\int_{X/S} \phi \bigwedge_{i=1}^n c_1(\bar{L}_i) \right).$$

The continuity of the fiber integral of C^∞ -forms in strong sense is well known (cf. [10, Theorem 3.8] and [7, Theorem 3.3.2]). Thus,

$$\int_{X'/S'} v^*(\phi) \bigwedge_{i=1}^n c_1(v^*(\bar{L}_i))$$

is continuous. Here u is projective and birational. In particular, u is continuous, closed and surjective. Hence, we can see that

$$\int_{X/S} \phi \bigwedge_{i=1}^n c_1(\bar{L}_i)$$

is continuous. □

3. PROOF OF THEOREM A

In this section, we will give the proof of Theorem A. We divide the proof of Theorem A into three steps.

Step 1. Let \bar{L} be a C^∞ -hermitian line bundle on X . If L is f -ample, then the metric of Deligne's pairing $\langle \bar{L}^{(n+1)} \rangle(X/S)$ is continuous.

Since $\langle \bar{L}^{\otimes m \cdot (n+1)} \rangle(X/S) = \langle \bar{L}^{(n+1)} \rangle(X/S)^{\otimes m^{n+1}}$, we may assume that L is f -very ample. Then, we have an embedding $\phi : X \hookrightarrow \mathbb{P}(f_*(L))$ over S with $\phi^*(\mathcal{O}_{\mathbb{P}(f_*(L))}(1)) = L$. Since our question is a local problem with respect to S , we may assume that $f_*(L) \simeq \mathcal{O}_S^{\oplus N+1}$ for some N . Thus, the above embedding gives rise to $\phi : X \hookrightarrow \mathbb{P}^N \times S$ over S . Here we give L a new metric $\|\cdot\|'$ arising from the standard Fubini-Study metric of $\mathcal{O}_{\mathbb{P}^N}(1)$. We denote this new C^∞ -hermitian line bundle by \bar{L}' .

We will show that the metric of $\langle \bar{L}^a, \bar{L}'^{n+1-a} \rangle(X/S)$ is continuous for every $0 \leq a \leq n+1$ by induction on a . If $a = 0$, then this holds by virtue of [11, Theorem 1.4, Theorem 1.6 and Theorem 3.6]. Denoting the metric of \bar{L} by $\|\cdot\|$, we set

$$u = \log \left(\frac{\|\cdot\|'}{\|\cdot\|} \right).$$

Then, u is a C^∞ -function on X . On the other hand,

$$\begin{aligned} \langle \bar{L}^a, \bar{L}'^{n+1-a} \rangle(X/S) \otimes \langle \bar{L}^{a+1}, \bar{L}'^{n-a} \rangle(X/S)^{\otimes -1} \\ = \langle \bar{L}^a, \bar{L}' \otimes \bar{L}^{\otimes -1}, \bar{L}'^{n-a} \rangle(X/S) = \langle \bar{L}^a, (\mathcal{O}_X, \exp(u)| \cdot |_X), \bar{L}'^{n-a} \rangle(X/S) \\ = \left(\mathcal{O}_S, \exp \left(\int_{X/S} u c_1(\bar{L})^a \wedge c_1(\bar{L}')^{\wedge n-a} \right) | \cdot |_S \right), \end{aligned}$$

where for an algebraic variety T over \mathbb{C} , we denote the canonical metric of \mathcal{O}_T by $| \cdot |_T$. It follows from Theorem 2.1 that $\int_{X/S} u c_1(\bar{L})^a \wedge c_1(\bar{L}')^{\wedge n-a}$ is continuous. Therefore, by hypothesis of induction, the metric of $\langle \bar{L}^{a+1}, \bar{L}'^{n-a} \rangle(X/S)$ is continuous.

Step 2. Let $\bar{L}_0, \dots, \bar{L}_n$ be C^∞ -hermitian line bundles on X . If L_0, \dots, L_n are f -ample, then the metric of Deligne's pairing $\langle \bar{L}_0, \dots, \bar{L}_n \rangle(X/S)$ is continuous.

It is not difficult to see that, for variables X_0, \dots, X_n ,

$$(3.5) \quad (n+1)! X_0 \cdots X_n = \sum_{I \subset \{0,1,\dots,n\}} (-1)^{n+1-\#(I)} \left(\sum_{i \in I} X_i \right)^{n+1}.$$

This formula shows us that

$$\langle \bar{L}_0, \dots, \bar{L}_n \rangle(X/S)^{\otimes (n+1)!} = \bigotimes_{I \subset \{0,1,\dots,n\}} \langle \bar{L}_I^{(n+1)} \rangle(X/S)^{\otimes (-1)^{n+1-\#(I)}},$$

where $\bar{L}_I = \bigotimes_{i \in I} \bar{L}_i$. Therefore, by Step 1, we can see that the metric of $\langle \bar{L}_0, \dots, \bar{L}_n \rangle(X/S)$ is continuous.

Step 3. General case.

Let \bar{L} be a C^∞ -hermitian line bundle on X such that L and $L \otimes L_i^{\otimes -1}$'s are f -ample. We set $\bar{M}_i = \bar{L} \otimes \bar{L}_i^{\otimes -1}$. Then,

$$\langle \bar{L}_0, \dots, \bar{L}_n \rangle(X/S) = \langle \bar{L} \otimes \bar{M}_0^{\otimes -1}, \dots, \bar{L} \otimes \bar{M}_n^{\otimes -1} \rangle(X/S).$$

Thus, using the linearity of Deligne's pairing and Step 2, we can conclude the proof of our theorem. \square

4. PROOF OF THEOREM B

Before starting the proof of Theorem B, let us begin with two lemmas.

Lemma 4.1. *Let X be a smooth projective curve of genus g over a field K . Let us consider Deligne's pairing*

$$\langle \cdot, \cdot \rangle : \text{Pic}(X \times J_X) \times \text{Pic}(X \times J_X) \rightarrow \text{Pic}(J_X)$$

with respect to the projection $X \times J_X \rightarrow J_X$, where $J_X = \text{Pic}^0(X)$. Then, we have the following.

- (1) A line bundle $\langle Q, Q \rangle$ does not depend on the choice of a universal line bundle Q on $X \times J_X$. In other words, if Q and Q' are universal line bundles on $X \times J_X$, then $\langle Q, Q \rangle \simeq \langle Q', Q' \rangle$.
- (2) We assume that there is $c_0 \in \text{Pic}^1(X)$ with $(2g-2)c_0 \sim c_1(\omega_X)$. (This holds if we enlarge the base field K .) Let us consider an embedding $\phi : X \rightarrow J_X$ given by $\phi(x) = x - c_0$. Let Θ_X be the theta divisor on J_X in terms of ϕ , namely, Θ_X is an ample and symmetric divisor on J_X defined by

$$\Theta_X = \{a \in J_X \mid a = \phi(x_1) + \dots + \phi(x_{g-1}) \text{ for some } x_1, \dots, x_{g-1} \in X\}.$$

If Q is a universal line bundle, then $\langle Q, Q \rangle \simeq \mathcal{O}_{J_X}(-2\Theta_X)$.

Proof. (1) Let $p_1 : X \times J_X \rightarrow X$ (resp. $p_2 : X \times J_X \rightarrow J_X$) be the projection onto the first (resp. second) factor. Then, there is a line bundle M on J_X with $Q' = Q \otimes p_2^*(M)$. Then,

$$\langle Q', Q' \rangle = \langle Q, Q \rangle \otimes \langle Q, p_2^*(M) \rangle^{\otimes 2} \otimes \langle p_2^*(M), p_2^*(M) \rangle.$$

Here $\langle Q, p_2^*(M) \rangle = \langle p_2^*(M), p_2^*(M) \rangle = \mathcal{O}_{J_X}$ because Q has the degree 0 along the fibers of p_2 . Thus, we get (1).

(2) We set

$$Q_X = \mathcal{O}_{X \times J_X} (p_1^* \phi^*(\Theta_X) + p_2^*(\Theta_X) - s^*(\Theta_X)),$$

where $s : X \times J_X \rightarrow J_X$ is a morphism given by $s(x, a) = \phi(x) + a$. Then, Q_X is a universal line bundle on $X \times J_X$.

First, we claim that $\langle Q_X, p_1^*(\omega_X) \rangle = \mathcal{O}_{J_X}$. It is easy to see that

$$\langle p_1^* \phi^*(\mathcal{O}_{J_X}(\Theta_X)), p_1^*(\omega_X) \rangle = \mathcal{O}_{J_X} \quad \text{and} \quad \langle p_2^*(\mathcal{O}_{J_X}(\Theta_X)), p_1^*(\omega_X) \rangle = \mathcal{O}_{J_X}((2g-2)\Theta_X)$$

On the other hand, if we set $c_1(\omega_X) \sim \sum_{i=1}^{2g-2} x_i$, then

$$\langle s^*(\mathcal{O}_{J_X}(\Theta_X)), p_1^*(\omega_X) \rangle = \mathcal{O}_{J_X} \left(\sum_{i=1}^{2g-2} T_{\phi(x_i)}^*(\Theta_X) \right),$$

where, for $a \in J_X$, $T_a : J_X \rightarrow J_X$ is a morphism given by $T_a(z) = z + a$. Thus, since $\phi(x_1) + \dots + \phi(x_g) = 0$,

$$\begin{aligned} \langle Q_X, p_1^*(\omega_X) \rangle &= \mathcal{O}_{J_X} \left(\sum_{i=1}^{2g-2} (\Theta_X - T_{\phi(x_i)}^*(\Theta_X)) \right) \\ &= \mathcal{O}_{J_X} \left(\Theta_X - T_{\phi(x_1) + \dots + \phi(x_{2g-2})}^*(\Theta_X) \right) \\ &= \mathcal{O}_{J_X}. \end{aligned}$$

Therefore, we get our claim.

Let us start the proof of (2). First of all, it is well known that $\det R(p_2)_*(Q_X) = \mathcal{O}_{J_X}(-\Theta_X)$. On the other hand, by Riemann-Roch theorem,

$$\det R(p_2)_*(Q_X)^{\otimes 2} = \langle Q_X, Q_X \rangle \otimes \langle Q_X, p_2^*(\omega_X) \rangle^{\otimes -1} \otimes \det R(p_2)_*(\mathcal{O}_{X \times J_X})^{\otimes 2}.$$

Here $\det R(p_2)_*(\mathcal{O}_{X \times J_X}) = \langle Q_X, p_2^*(\omega_X) \rangle = \mathcal{O}_{J_X}$. Thus, $\langle Q_X, Q_X \rangle = \mathcal{O}_{J_X}(-2\Theta_X)$. Therefore, by (1), we have (2). \square

Lemma 4.2. *Let $A \subseteq R$ be commutative rings such that R is flat over A . If A and the localization R_S with respect to a multiplicative subset S of $A \setminus \{0\}$ are integral domains, then R is also an integral domain.*

Proof. It is sufficient to show that the natural homomorphism $R \rightarrow R_S$ is injective. Pick up an element $a \in R$ with $a = 0$ in R_S . Then, there is $s \in S$ with $sa = 0$ in R . Since $s \neq 0$ and A is an integral domain, $A \xrightarrow{\times s} A$ is injective. Thus, so is $R \xrightarrow{\times s} R$ because R is flat over A . Therefore, $a = 0$ in R . \square

Let us start the proof of Theorem B. Fix a universal line bundle Q on $X \times J_X$. Then, we can find a morphism $q : \mathcal{Y} \rightarrow \mathcal{J}$ of projective arithmetic varieties over B and a C^∞ -hermitian line bundle \overline{Q} on \mathcal{Y} such that $q : \mathcal{Y} \rightarrow \mathcal{J}$ coincides with the projection $X \times J_X \rightarrow J_X$ over K , and that \overline{Q} is equal to Q over K . If q is not flat, taking Raynaud's flattening theorem [9], we may assume that q is flat. Note that after flattening, \mathcal{Y} is still integral by Lemma 4.2. Let B_m be the closure of the point $[L^{\otimes m}] \in J_X$ in \mathcal{J} . We set

$$\mathcal{X}_m = q^{-1}(B_m) \quad \text{and} \quad \overline{\mathcal{L}}_m = \left(\overline{Q}|_{\mathcal{X}_m} \right)^{\otimes 1/m}$$

as an element of $\widehat{\text{Pic}}(\mathcal{X}_m) \otimes \mathbb{Q}$. Then, since \mathcal{X}_m is integral by Lemma 4.2, we can see that $(\mathcal{X}_m, \overline{\mathcal{L}}_m)$ is a C^∞ -model of (X, L) . Moreover, $\mathcal{X}_m \rightarrow B_m$ is flat, and $\pi_m : B_m \rightarrow B$ is birational. Here, since Deligne's pairing is compatible with base changes,

$$\langle \overline{\mathcal{L}}_m^{\otimes m}, \overline{\mathcal{L}}_m^{\otimes m} \rangle(\mathcal{X}_m/B_m) = \langle \overline{Q}, \overline{Q} \rangle(\mathcal{Y}/\mathcal{J})|_{B_m}.$$

Thus,

$$\widehat{\deg} \left(\widehat{c}_1 \left(\langle \overline{\mathcal{L}}_m^{\otimes m}, \overline{\mathcal{L}}_m^{\otimes m} \rangle(\mathcal{X}_m/B_m) \right) \cdot \widehat{c}_1(\pi_m^*(\overline{H}))^d \right) = h_{\langle \overline{Q}, \overline{Q} \rangle(\mathcal{Y}/\mathcal{J})}^{\overline{B}}([L^{\otimes m}]).$$

Clearly,

$$\widehat{\deg} \left(\widehat{c}_1 \left(\langle \overline{\mathcal{L}}_m^{\otimes m}, \overline{\mathcal{L}}_m^{\otimes m} \rangle(\mathcal{X}_m/B_m) \right) \cdot \widehat{c}_1(\pi_m^*(\overline{H}))^d \right) = m^2 \widehat{\deg} \left(\widehat{c}_1(\overline{\mathcal{L}}_m) \cdot \widehat{c}_1(\overline{\mathcal{L}}_m) \cdot \widehat{c}_1(f_m^*(\overline{H}))^d \right),$$

where f_m is the composition of morphisms $\mathcal{X}_m \rightarrow B_m \rightarrow B$. Hence, we get

$$\widehat{\deg}(\hat{c}_1(\overline{\mathcal{L}}_m) \cdot \hat{c}_1(\overline{\mathcal{L}}_m) \cdot \hat{c}_1(f_m^*(\overline{H}))^d) = \frac{1}{m^2} h_{\langle \overline{\mathcal{Q}}, \overline{\mathcal{Q}} \rangle(\mathcal{Y}/\mathcal{J})}^{\overline{B}}([L^{\otimes m}]).$$

On the other hand, the metric of $\langle \overline{\mathcal{Q}}, \overline{\mathcal{Q}} \rangle(\mathcal{Y}/\mathcal{J})$ is continuous by Theorem A. Thus, by [8, Corollary 3.3.5] and Lemma 4.1, we can see that

$$\lim_{n \rightarrow \infty} \frac{1}{m^2} h_{\langle \overline{\mathcal{Q}}, \overline{\mathcal{Q}} \rangle(\mathcal{Y}/\mathcal{J})}^{\overline{B}}([L^{\otimes m}]) = \hat{h}_{-2\Theta_X}^{\overline{B}}([L]) = -2\hat{h}_{\Theta_X}^{\overline{B}}([L]).$$

Therefore, we get our theorem.

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